Periodic points of Logistic map with diffusion

Akihiko Itaya and Seiji Ukai
Department of Mathematical and Computing Science
Tokyo Institute of Technology
2-12-1, Ookayama Meguro-ku, Tokyo 152, Japan

Abstract

In this paper, we show that 'Logistic map with diffusion' has infinitely many periodic points of all period n $(n = 1, 2, \cdots)$. This map is obtained by adding the diffusion term to the well known Logistic equation and by discretising it in time semi-implicitly, and thus can serve as a nice example of the infinite-dimensional map which has periodic points of arbitrary high periods. Our proof is based on a somewhat general theorem on the existence of periodic points which can be applied to both multi-dimensional and infinite-dimensional maps.

1. Introduction and main result.

The periodic points of maps have been studied by many authors, mainly in the context of chaotic behaviors of discrete dynamical system. In particular, the theory for one-dimensional maps has been fully developed with a number of deep and elegant results, see [3] [6] [9] [10] [12] and the references therein. There are also many satisfactory results on periodic points of multi-dimensional maps, [11] [13] [14].

However, most of periodic points found so far for infinite-dimensional maps are those of low periods and few example are known which have periodic points of arbitrarily high periods. In this paper, we will present a simple example of the infinite-dimensional map which has infinitely many periodic points of all period $n = 1, 2, \cdots$. It is the "Logistic map with diffusion". This map exhibits chaotic characters as well, which will be discussed in a subsequent paper.

Recall the famous one-dimensional Logistic map

$$x_{n+1} = \lambda x_n(1 - x_n) \tag{1.1}$$

which can be obtained by Euler's difference scheme for the Logistic equation

$$\frac{dv}{dt} = \lambda v(1 - v), \qquad (v \in \boldsymbol{R}). \tag{1.2}$$

Now consider a P.D.E which is obtained by adding the diffusion term to (1.2),

$$\frac{\partial v}{\partial t} = \alpha \Delta v + \lambda v(1 - v), \tag{1.3}$$

where $\triangle$ is the Laplacian in $\boldsymbol{R}^d$ and $\alpha > 0$ is the diffusion coefficient. The equation of this type is usually called Fisher's equation. Discretize (1.3) semi-implicitly in t, in the manner,

$$\frac{v_{n+1} - v_n}{\Delta t} = \alpha \triangle v_{n+1} + \lambda v_n (1 - v_n). \tag{1.4}$$

Then, we obtain a discrete model for (1.3);

$$(r - \triangle) u_{n+1} = \mu u_n (1 - u_n), \tag{1.5}$$

where

$$u_n = \frac{\lambda \Delta t}{1 + \lambda \Delta t} v_n,$$

$$\mu = \frac{1 + \lambda \Delta t}{\alpha \Delta t},$$

and

$$r = \frac{1}{\alpha \Delta t}.$$

If we consider (1.5) in a domain $D \subset \boldsymbol{R}^d$ with an appropriate boundary condition, then it is rewritten as

$$u_{n+1} = F(u_n), \tag{1.6}$$

where

$$F(u) = (r - \triangle)^{-1} \mu h(u), \qquad h(u) = u(1 - u). \tag{1.7}$$

Here and hereafter, we call this map F the "Logistic map with diffusion". Note that in the case of the Neumann boundary condition, (1.6) has constant solutions, and hence can be reduced to the one-dimensional map (1.1). Otherwise, F is essentially an infinite-dimensional map and we cannot conclude immediately that (1.6) has the same character as (1.1). However, a numerical computation on (1.5) with $D = [0, 1]$ and with the Dirichlet boundary condition by discretizing the Laplacian $\triangle$ in the central difference shows that (1.5) has a bifurcation diagram quite similar to that of (1.1). See Figure 1.1. Moreover, this diagram does not change essentially with the mesh size. It seems therefore that (1.5) with the Dirichlet boundary condition also produces infinitely many cascades of period doubling bifurcations and chaos, just like (1.1).

To prove this was our original motivation of the present work, but is yet an open problem. Instead, in this paper, we show that (1.5) has all of n-periodic points ($n = 1, 2, \cdots$) under the boundary condition

$$(1 - \theta) \frac{\partial u}{\partial \nu} + \theta u = 0, \qquad (x \in \partial D) \tag{1.8}$$

where ν is the unit outward normal to ∂D and $\theta = \theta(x)$ is some given function on ∂D. The main result of this paper is,

Theorem 1.1. *Suppose that $D \subset \boldsymbol{R}^d$ is a bounded domain with smooth boundary ∂D and that $\theta = \theta(x) \in [0, 1]$ is smooth on ∂D. Then there exists a number $\delta > 0$ satisfying the following. If $r + \| \theta \|_{L^\infty(D)} < \delta$, then there exist positive numbers μ_1, μ_2 $(0 < \mu_1 < \mu_2)$ such that the map (1.6) associated with the boundary condition (1.8) has infinitely many periodic points of all periods $n = 1, 2, \cdots$, for all $\mu \in [\mu_1, \mu_2]$.*

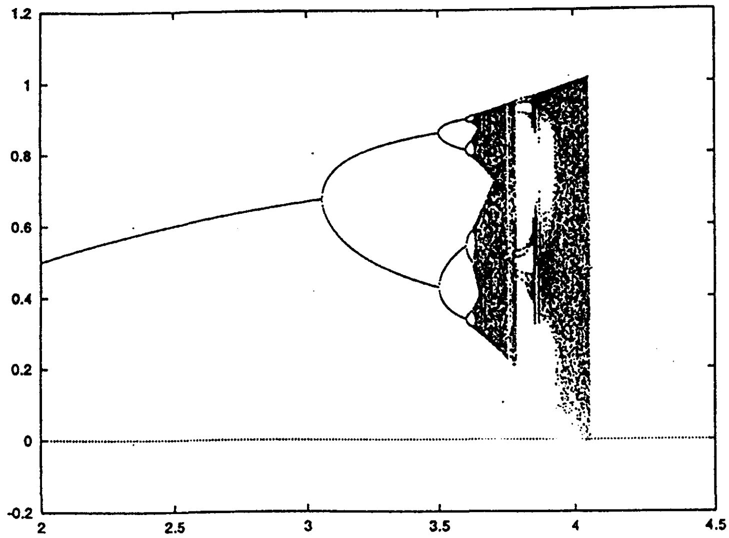


Figure 1.1

Remark 1.1. That u is a periodic point of F of period n means that

$$\begin{cases} u = F^n(u), \\ u \neq F^k(u), \qquad k = 1, 2, \cdots, n-1, \end{cases} \tag{1.9}$$

where F^n denotes the n-th iteration of the map F. In view of (1.5), (1.9) is equivalent to the set of partial differential equations,

$$\begin{cases} (r - \Delta)u_1 = \mu h(u_2), \\ (r - \Delta)u_2 = \mu h(u_3), \\ \qquad \vdots \\ \qquad \vdots \\ (r - \Delta)u_n = \mu h(u_1), \\ u_i \neq u_j \qquad (i \neq j), \end{cases} \tag{1.10}$$

with the boundary condition (1.8) on u_i's. Notice that it is easy to see that F^n has a fixed point u. Thus, the crucial part in the proof is the part "$u \neq F^k(u)$ for all $k = 1, 2, \cdots, n-1$". On the other hand, in our proof of Theorem 1.1, it is not necessary that r, θ and h are the same for all the equations in (1.10).

Remark 1.2. In our theorem, r must be small, that is, either α or Δt must be large. We also need to require that θ is small. That is, the boundary condition (1.8) must be close to the Neumann boundary condition. In particular, the case of the Dirichlet boundary condition ($\theta = 1$) is not covered by Theorem 1.1. However, our numerical computation implies that the same conclusion should be true also under the Dirichlet boundary condition.

Remark 1.3. It is possible to show that F has a non-trivial periodic point of period 1 (i.e fixed point of F) for all $\mu \in [\mu_1, \infty)$ with some $\mu_1 > 0$, for all the boundary condition including the Dirichlet boundary condition.
Remark 1.4. In our theorem, the upper bound μ_2 of μ for which infinitely many periodic points exist is finite. On the other hand, it is well-known that $\mu_2 = \infty$ for the one-dimensional unimodal map like (1.1), and hence one might expect that this would be the case also in the multi-dimensional case. However, there is an example of two-dimensional unimodal map for which $\mu_2 < +\infty$. Consider the two-dimensional map,

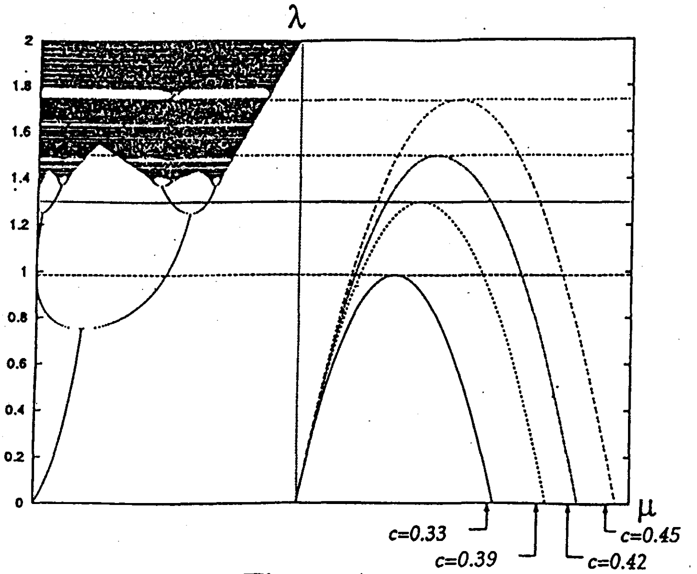
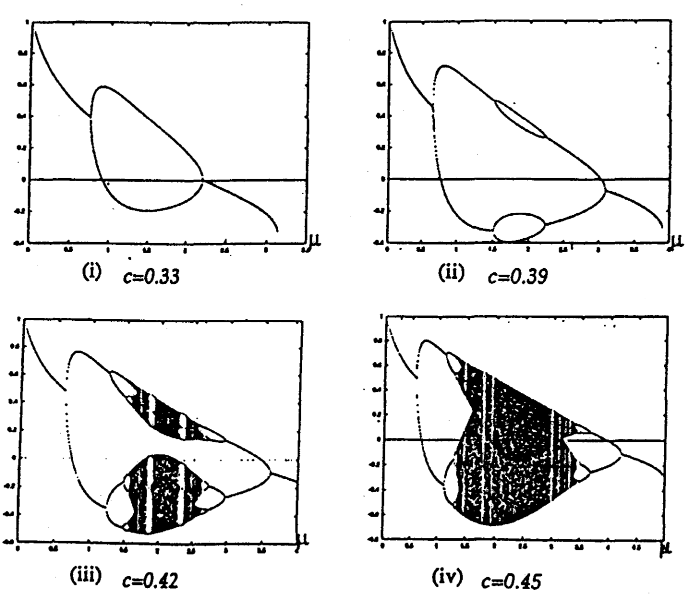


Figure 1.2



(i) c=0.33 (ii) c=0.39

(iii) c=0.42 (iv) c=0.45

Figure 1.3

$$(1.11)\qquad F(x,y)=(1-\mu(x^2+y^2),\ a-\mu c(x^2+y^2)),\qquad (x,y\in \boldsymbol{R},\quad a,\ c>0),$$

which may be taken as a simple two-dimensional version of the one-dimensional map,

$$(1.12)\qquad f(x)=1-\mu x^2,\qquad (x\in \boldsymbol{R}).$$

(1.12) is equivalent to (1.1) while (1.11) arises if we discretize the Laplacian $\triangle$ of (1.5) on $D=[0,1]$ with $\theta=1$ only with 5 mesh points and then modify slightly some coefficients in the resulting map. To see that $\mu_2<+\infty$, it suffices to note that (1.11) has the one-dimensional invariant manifold,

$$(1.13)\qquad cx-y=c-a,$$

which is globally stable for F, that is, all points (x,y) in $\boldsymbol{R}^2$ are mapped onto the line (1.13) by F, so that F is equivalent to the one-dimensional map,

$$(1.14)\qquad g(u)=1-\lambda(\mu)u^2,\qquad (u\in \boldsymbol{R}),$$

where

$$(1.15)\qquad \lambda(\mu)=\mu((1+ac)-(a-c)^2\mu).$$

(1.15) implies that the folding of the parameter μ occurs and that the bifurcation diagram changes along with the values of a and c. Figure 1.2 shows the relation between (1.15) and the bifurcation diagram of (1.14) with $a=1$. The left part of Figure 1.2 is the bifurcation diagram of (1.14) when λ is taken independent of μ, where the vertical axis is for λ and the horizontal axis for u. On the other hand, the right part of Figure 1.2 shows the graphs of (1.15) for $c=0.33$. 0.39, 0.42 and 0.45 respectively, where the vertical axis is for λ and the horizontal axis for μ. From Figure 1.2, we can see that if $c=0.33$, $\lambda(\mu)$ is less than 1 for all $\mu\geq 0$, that is, (1.14) has periodic points of period one and two only. Actually, the bifurcation diagram of (1.11) with $c=0.33$ looks like Figure 1.3 (i). Similar reasoning applies with other values of c and gives Figure 1.3 (ii)–(iv), showing that $\mu_2<+\infty$.

Clearly, our map F has not a one-dimensional invariant manifold, expect for the Neumann boundary condition, but since Figure 1.1 looks very similar to the bifurcation diagram of the one-dimensional map (1.1), it is natural to expect that F has an one-(or low-)dimensional "approximate" invariant manifold. Thus, suggested by Figure 1.1, we are led to the following setting for proving Theorem 1.1.

Let Ω be a non-empty bounded, closed convex subset of a Banach space Y and set $U=\boldsymbol{R}\times\Omega$ with $x\in \boldsymbol{R}$, $y\in\Omega$. We consider a map $F=(f,g)$ with $f:U\to\boldsymbol{R}$, $g:U\to Y$ satisfying the following. See Figure 1.4.

(F0) $f, f_x\in C(U,\boldsymbol{R})$.

(F1) There exists an interval $I=[a,b]$, $(0\leq a<b)$, such that

- **(i)** $f(a,y)\geq b,\quad f(b,y)\leq a,\qquad (\forall y\in\Omega)$,
- **(ii)** $f_x(x,y)\leq 0,\qquad (\forall x\in I, y\in\Omega)$.

(F2) There exists an interval $\tilde{I}=[\tilde{b},\tilde{a}]$, $(\tilde{b}<\tilde{a}\leq 0)$, such that

(i) $f(\tilde{a}, y) \geq b, \quad f(\tilde{b}, y) \leq a, \qquad (\forall y \in \Omega),$
(ii) $f_x(x, y) \geq 0, \qquad (\forall x \in \tilde{I}, y \in \Omega).$

It will be shown in §5 that under these assumptions,

$$\exists 1\ \tilde{\varphi} \in C(\Omega, \tilde{I}), \quad f(\tilde{\varphi}(y), y) = a, \qquad (\forall y \in \Omega). \tag{1.16}$$

Since $\tilde{\varphi}(y) \in \tilde{I}$ $(\forall y \in \Omega)$, there is a $\tilde{\varphi}_* \in \tilde{I}$ such that

$$\tilde{\varphi}_* = \inf\{\tilde{\varphi}(y) \mid y \in \Omega\}. \tag{1.17}$$

We shall further assume
(F3) $f(b, y) \leq \tilde{\varphi}_*$,
(F4) $g : (\tilde{I} \cup I) \times \Omega \to \Omega$ is compact.
Here and hereafter a map is said compact if it is continuous and if the image of any bounded set is pre-compact. We can now state a rather general theorem for the existence of periodic points, which can be applied to our map (1.6).

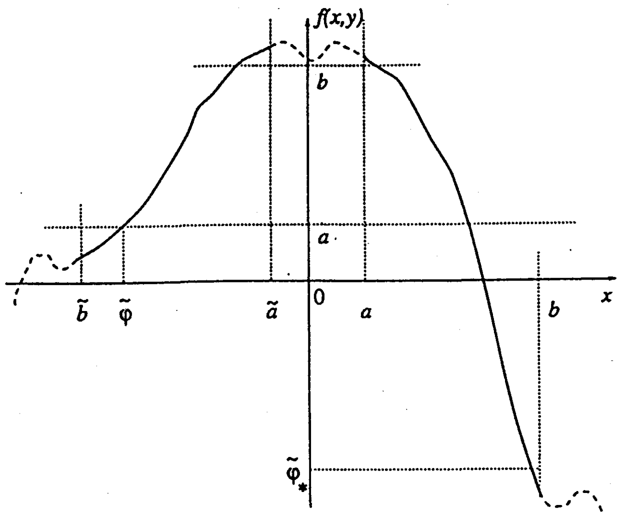


Figure 1.4

Theorem 1.2. *Under the conditions* **(F0)–(F4)**, *F has infinitely many periodic points of all periods $n = 1, 2, \cdots$.*

The plan of this paper is as follows. In the next section, we will give some preliminaries and make a reduction of our problem. The proof of Theorem 1.1 is given in §3 by showing that Theorem 1.2 can be applied to the reduced system given in §2, and Theorem 1.2 will be proved in §4. §5 is devoted to the proofs of (1.16) and Proposition 4.1 which are essentially used in §4.

2. Preliminaries and reduction of problem.

Throughout this paper, $\| \cdot \|_p$ will denote the norm of $L^p(D)$ $(1 \leq p \leq \infty)$, $< \cdot, \cdot >$ the inner product of $L^2(D)$, and $W^{k,p}(D)$ the usual L^p-Sobolev space of order k.

Recall that D is a bounded domain of $\boldsymbol{R}^d$ with the smooth boundary ∂D and $\theta = \theta(x) \in [0,1]$ is a smooth function on ∂D. First, we consider the eigenvalue problem,

$$\text{(2.1)} \qquad \begin{cases} (r - \Delta)\phi = \kappa\phi, & (x \in D), \\ (1-\theta)\dfrac{\partial \phi}{\partial \nu} + \theta\phi = 0, & (x \in \partial D), \end{cases}$$

Let κ_n be the n-th eigenvalue of the above problem and ϕ_n be a normalized (in $L^2(D)$) eigenfunction corresponding to κ_n. Since we are assuming $r > 0$ and $0 \leq \theta(x) \leq 1$,

$$0 < \kappa_1 < \kappa_2 \leq \kappa_3 \leq \cdots$$

and $\kappa_n \to \infty$ when $n \to \infty$. Note that $\kappa_1 = r$ under the Neumann boundary condition $(\theta = 0)$. It is well known that $\{\phi_n\}$ is a complete orthonormal system of $L^2(D)$. In the following, we need the

Lemma 2.1. *$\kappa_1 \to 0$ and $\kappa_2 \geq c_0$ with some constant $c_0 > 0$, as $r + \| \theta \|_\infty \to 0$.*

Proof. This fact is rather well-known, but we will give a proof for the sake of completeness. Put

$$\beta(x) = \frac{\theta(x)}{1 - \theta(x)}.$$

According to the max-min principle [4], the eigenvalue κ_n is characterized by

$$\text{(2.2)} \qquad \kappa_n = \sup_{\substack{E \\ dimE=n}} \inf_{\substack{u \in E^\perp \\ \|u\|_2 = 1}} \left\{ r \| u \|_2^2 + \| \nabla u \|_2^2 + \int_{\partial D} \beta(x) \mid u(x) \mid^2 d\sigma \right\}$$

where E denotes a finite dimensional subspace of $L^2(D)$ and $d\sigma$ is the usual surface measure on ∂D. Since $\beta(x) \geq 0$ when $0 \leq \theta(x) < 1$, we have

$$\begin{aligned} \kappa_n &\geq \sup_{\substack{E \\ dimE=n}} \inf_{\substack{u \in E^\perp \\ \|u\|_2 = 1}} \left\{ r \| u \|_2^2 + \| \nabla u \|_2^2 \right\} \\ &= r + \eta_n \end{aligned}$$

where

$$\eta_n = \sup_{\substack{E \\ dimE=n}} \inf_{\substack{u \in E^\perp \\ \|u\|_2 = 1}} \| \nabla u \|_2^2$$

is nothing but the n-th eigenvalue of the Laplacian with the Neumann boundary condition, in view of the same characterization by the max-min principle. Clearly $\eta_1 = 0$ and $\eta_2 > 0$. Take $c_0 = \eta_2$ and thus $\kappa_2 \geq c_0$ for any r and θ. On the other hand, by the trace theorem [4],

$$\begin{aligned} \int_{\partial D} \beta(x) \mid u(x) \mid^2 d\sigma &\leq \| \beta \|_\infty \int_{\partial D} \mid u(x) \mid^2 d\sigma \\ &\leq c_1 \| \beta \|_\infty \| u \|_{H^1(D)}^2 \end{aligned}$$

with some constant $c_1 > 0$ independent of u. Since $\| u \|_{H^1(D)}^2 = \| u \|_2^2 + \| \nabla u \|_2^2$, (2.2) gives

$$\begin{aligned}
\kappa_n &\leq \sup_{\substack{E \\ dimE=n}} \inf_{\substack{u\in E^\perp \\ \|u\|_2=1}} \left\{ (r + c_1 \| \beta \|_\infty) \| u \|_2^2 + (1 + c_1 \| \beta \|_\infty) \| \nabla u \|_2^2 \right\} \\
&= r + c_1 \| \beta \|_\infty + (1 + c_1 \| \beta \|_\infty)\eta_n,
\end{aligned}$$

whence $\kappa_1 \leq r + c_1 \| \beta \|_\infty$. This completes the proof of the lemma. ∎

Define the subspace E_k of $L^2(D)$ and the orthogonal projection P_k by

$$\begin{aligned}
E_k &= span\{\phi_1, \phi_2, \cdots, \phi_k\}, \\
P_k &: L^2(D) \to E_k,
\end{aligned}$$

for $k = 1, 2, \cdots$. For the convenience, we set $E_0^\perp = L^2(D)$ and $P_0^\perp = Id$. Associated with the boundary condition (1.8), we consider the Green operator

$$G = (r - \Delta)^{-1}.$$

Lemma 2.2. *$P_k^\perp G : L^2(D) \to E_k^\perp$ is compact with*

$$\| P_k^\perp G v \|_2 \leq \frac{1}{\kappa_{k+1}} \| v \|_2, \quad \forall v \in L^2(D).$$

for $k = 0, 1, 2, \cdots$.

This is well-known and the proof is omitted.

In order to deal with the nonlinear term $h(u)$, we need L^p-estimates of G. Put $u = Gv$, which solves

$$\begin{cases} (r - \Delta)u = v, & in \quad D, \\ (1-\theta)\dfrac{\partial u}{\partial \nu} + \theta u = 0, & on \quad \partial D. \end{cases} \tag{2.3}$$

The celebrated L^p-theory of elliptic equations [2] says that for all $p \in (1, \infty)$ and $k = 0, 1, 2, \cdots$,

$$\| u \|_{W^{k+2,p}(D)} \leq M_1(\| v \|_{W^{k,p}(D)} + \| u \|_p) \tag{2.4}$$

holds with a constant $M_1 > 0$ depending on r, θ, D, p, k but not on v. More precisely,

$$M_1 = M_1(r_0, \Theta, D, p, k) \tag{2.5}$$

for all $| r | \leq r_0$ and $\theta \in \Theta$ where $r_0 > 0$ is any number and Θ is any bounded set of $C^{k+2}(\partial D)$ and $M_1 > 0$ depends only on r_0, Θ, D, p and k. A simple application of this is

Lemma 2.3. *$\phi_1 \in C(\overline{D}) \cap L^p(D)$ for all $p \geq 2$ with*

$$M_2 \leq \| \phi_1 \|_p \leq M_3 \tag{2.6}$$

for all $r \in (0, r_0]$ and $\theta \in \Theta$ where $M_2 = | D |^{1-p/2}$ and $M_3 > 0$ is a constant depending only on $r_0 > 0$, Θ and D.

Proof. We know $\phi_1 \in L^2(D)$ with $\| \phi_1 \|_2 = 1$. The lower bounds comes from Hölder's inequality

$$1 = \| \phi_1 \|_2 \leq | D |^{1-\frac{2}{p}} \| \phi_1 \|_p^{\frac{2}{p}} .$$

To prove that $\phi_1 \in C(\overline{D})$, we apply (2.4) for $p = 2$ to (2.1), to get

$$\| \phi_1 \|_{W^{k+2,2}(D)} \leq M_1 \{ \kappa_1 \| \phi_1 \|_{W^{k,2}(D)} + \| \phi_1 \|_2 \}$$

for all $r > 0$. Iterate this in $k = 0, 2, \cdots$, and obtain for each even integer $l > d/2$

$$\| \phi_1 \|_{W^{l,2}(D)} \leq C M_1 \| \phi_1 \|_2$$

which shows $\phi_1 \in C(\overline{D})$ and (2.6), thanks to the Sobolev embedding theorem. [1] ■

Now we can prove the

Lemma 2.4. *Let $p > d/2$. Then $P_1^{\perp} G : L^p(D) \cap L^2(D) \to C(\overline{D})$ is compact with*

$$\| P_1^{\perp} G v \|_{\infty} \leq M_4 (\| P_1^{\perp} v \|_p + \| P_1^{\perp} v \|_2) \tag{2.7}$$

for all $r \in (0, r_0]$ and $\theta \in \Theta$ where $M_4 = M_4(r_0, \Theta, D, p)$.

Proof. Set $w = P_1^{\perp} G v = G P_1^{\perp} v = P_1^{\perp} G P_1^{\perp} v$. In view of Lemma 2.2, we get

$$\| w \|_2 \leq \frac{1}{\kappa_2} \| P_1^{\perp} v \|_2 . \tag{2.8}$$

On the other hand, w solves (2.3) with $P_1^{\perp} v$ in place of v. Apply (2.4) for $k = 0$ and $p = 2$ to deduce

$$\begin{aligned} \| w \|_{W^{2,2}(D)} &\leq M_1 (\| P_1^{\perp} v \|_2 + \| w \|_2) \\ &\leq M_1 (1 + \frac{1}{\kappa_2}) \| P_1^{\perp} v \|_2 \qquad (by \quad (2.8)). \end{aligned}$$

Note that by Lemma 2.1, $\kappa_2 \geq c_0 > 0$, for all $r > 0$ and θ. Owing to the Sobolev embedding theorem, we are done for $d = 1, 2, 3$, while for $d \geq 4$, we get

$$\| w \|_q \leq C \| w \|_{W^{2,2}(D)} \leq M_5 \| P_1^{\perp} v \|_2,$$

with any $q \in (2, \frac{2d}{d-4})$. With such a q, (2.4) yields

$$\begin{aligned} \| w \|_{W^{2,q}(D)} &\leq M_1 (\| P_1^{\perp} v \|_q + \| w \|_q) \\ &\leq M_1 (\| P_1^{\perp} v \|_q + M_5 \| P_1^{\perp} v \|_2). \end{aligned}$$

Proceeding as before, we then reach (2.7). ■

Now, we will make some reduction of our problem. Consider the dynamical system,

$$u' = F(u) = \mu G h(u). \tag{2.9}$$

and decompose u as

$$u = P_1 u + P_1^{\perp} u = z \phi_1 + y$$

where $z = < u, \phi_1 > \in \boldsymbol{R}$ and $y = P_1^{\perp} u \in E_1^{\perp}$, and similarly for $u' = z'\phi_1 + y'$. Then (2.9) is decomposed as

$$
\left\{\begin{array}{lcl} z' & = & \lambda(z - \alpha_0 z^2 - 2 < y, \phi_1^2 > z - < y^2, \phi_1 >), \\ y' & = & \kappa_1 \lambda P_1^{\perp} G h(u), \end{array}\right. \tag{2.10}
$$

where we have put

$$
\begin{array}{lcl} \lambda & = & \dfrac{\mu}{\kappa_1}, \\ \alpha_0 & = & < \phi_1^2, \phi_1 > = \| \phi_1 \|_3^3 . \end{array}
$$

Furthermore, set

$$
\begin{array}{lcl} m & = & \dfrac{\lambda - 2}{4\alpha_0}, \\ v & = & \dfrac{1}{m}(z - \dfrac{1}{2\alpha_0}). \end{array}
$$

Then (2.10) can be reduced to

$$
\left\{\begin{array}{lcl} v' & = & 1 - \eta v^2 - \Phi(v, y), \\ y' & = & \kappa_1 \lambda P_1^{\perp} G h(u), \end{array}\right. \tag{2.11}
$$

where

$$
\eta = \eta(\lambda) = \lambda m \alpha_0 = \frac{\lambda(\lambda - 2)}{4}
$$

and

$$
\Phi(v, y) = \lambda\{2 < y, \phi_1^2 > v + \frac{1}{m\alpha_0} < y, \phi_1^2 > + \frac{1}{m} < y^2, \phi_1 >\}. \tag{2.12}
$$

3. Proof of Theorem 1.1.

In this section, we will prove Theorem 1.1 admitting Theorem 1.2. First, we set

$$
Y = E_1^{\perp} \cap C(\overline{D}) = \{y = u - < u, \phi_1 > \phi_1 : u \in L^2(D) \cap C(\overline{D})\}.
$$

By Lemma 2.3, we see that $Y \subset L^2(D) \cap C(\overline{D})$. We shall take

$$
\Omega = \{y \in Y : \| y \|_2 \leq \epsilon, \quad \| y \|_\infty \leq K\}
$$

where $\epsilon, K > 0$ are constants to be determined later. Following (2.11), we define two maps $f : \boldsymbol{R} \times \Omega \to \boldsymbol{R}$ and $g : \boldsymbol{R} \times \Omega \to Y$ as follows;

$$
\left\{\begin{array}{lcl} f(v, y) & = & 1 - \eta v^2 - \Phi(v, y), \\ g(v, y) & = & \kappa_1 \lambda P_1^{\perp} G h(u), \end{array}\right. \tag{3.1}
$$

where $v \in \boldsymbol{R}$ and $y \in \Omega$. The main goal is to show that Theorem 1.2 applies to (3.1) if $r + \| \theta \|_\infty$ is small.

It is clear that f satisfies the condition **(F0)**. Since

$$f_v(v,y) = -2\eta v - 2\lambda < y, \phi_1^2 >$$

and $|< y, \phi_1^2 >| \leq \| y \|_2 \| \phi_1 \|_2 \| \phi_1 \|_\infty \leq \| \phi_1 \|_\infty \epsilon$, we see that, if $\eta > 0$ $(\lambda > 2)$,

$$f_v \leq 0, \qquad \forall v \geq \frac{\lambda \| \phi_1 \|_\infty}{\eta}\epsilon, \quad \forall y \in \Omega,$$

and

$$f_v \geq 0, \qquad \forall v \leq -\frac{\lambda \| \phi_1 \|_\infty}{\eta}\epsilon, \quad \forall y \in \Omega,$$

so that **(F1)(ii)** and **(F2)(ii)** are satisfied if we take

$$a = \frac{\lambda \| \phi_1 \|_\infty}{\eta}\epsilon = \frac{\| \phi_1 \|_\infty}{m\alpha_0}\epsilon = \frac{4 \| \phi_1 \|_\infty}{\lambda - 2}\epsilon \qquad \textit{and} \qquad \tilde{a} = -a.$$

Note from Lemma 2.3 that there are constants $\lambda_0 > 2$ and $a_0 > 0$ such that for all $\lambda \geq \lambda_0$ and $\epsilon > 0$,

(3.2) $$a \leq a_0 \epsilon.$$

Since $|< y^2, \phi_1 >| \leq \| \phi_1 \|_\infty \epsilon^2$ and $\alpha_0 = \| \phi_1 \|_3^3 \leq \| \phi_1 \|_\infty$, we get

$$\begin{aligned} | \Phi(\pm a, y) | &\leq \lambda \| \phi_1 \|_\infty \{2a + \frac{1}{m\alpha_0} + \frac{1}{m}\epsilon\}\epsilon \\ &\leq \frac{\lambda \| \phi_1 \|_\infty}{m\alpha_0}\{3 \| \phi_1 \|_\infty \epsilon + 1\}\epsilon \end{aligned}$$

Then it holds that, for all $y \in \Omega$

$$\begin{aligned} f(\pm a, y) &\geq 1 - \eta a^2 - \frac{\lambda \| \phi_1 \|_\infty}{m\alpha_0}\{3 \| \phi_1 \|_\infty \epsilon + 1\}\epsilon \\ &= 1 - \frac{4\lambda \| \phi_1 \|_\infty}{\lambda - 2}(4 \| \phi_1 \|_\infty \epsilon + 1)\epsilon. \end{aligned}$$

Set

$$b = 1 - \rho.$$

where

$$\rho = \rho(\epsilon) = \frac{4\lambda \| \phi_1 \|_\infty}{\lambda - 2}(4 \| \phi_1 \|_\infty \epsilon + 1)\epsilon.$$

Clearly by Lemma 2.3, there is a constant $\rho_0 > 0$ such that for all $\lambda \geq \lambda_0$ and $\epsilon > 0$,

(3.3) $$\rho \leq \rho_0 \epsilon.$$

We must require $a < b$, and this is possible if

(3.4) $$\epsilon < \frac{1}{a_0 + \rho_0}.$$

Now, we get

$$
\begin{aligned}
|\,\Phi(\pm b,y)\,| &\le \lambda \parallel \phi_1 \parallel_\infty \{2\,|\,b\,| + \frac{1}{m\alpha_0} + \frac{1}{m}\epsilon\}\epsilon, \\
&\le 2\lambda \parallel \phi_1 \parallel_\infty \epsilon + \xi,
\end{aligned}
$$

where

$$
\xi = \xi(\epsilon) = \frac{4\lambda \parallel \phi_1 \parallel_\infty}{\lambda - 2}(1 + \alpha_0\epsilon)\epsilon.
$$

Again by Lemma2.3, there is a constant $\xi_0 > 0$ such that for all $\lambda \ge \lambda_0$ and $\epsilon > 0$,

$$
\xi \le \xi_0\epsilon.
$$

Then, we have, for $\lambda > \lambda_0$,

$$
\begin{aligned}
f(\pm b,y) &= 1 - \eta b^2 - \Phi(\pm b,y) \\
&\le 1 - \frac{\lambda(\lambda-2)}{4}(1-\rho_0\epsilon)^2 + 2\lambda \parallel \phi_1 \parallel_\infty \epsilon + \xi_0\epsilon, \\
&\le 1 - \frac{1}{4}(1-\rho_0\epsilon)^2\lambda^2 + \frac{1}{2}\{(1-\rho_0\epsilon)^2 + 4 \parallel \phi_1 \parallel_\infty \epsilon\}\lambda + \xi_0\epsilon.
\end{aligned}
$$

This implies that there is a number $\lambda' > \lambda_0$ such that $f(\pm b,y)$ is strictly decreasing on λ for all $\lambda \ge \lambda'$ and tend to $-\infty$ as $\lambda \to \infty$. Therefore, setting $\tilde{b} = -b$, we see that there exists a number $\lambda'' > \lambda'$ such that $f(b,y) < a$ and $f(\tilde{b},y) < a$ for all $\lambda \ge \lambda''$, thus f satisfies the conditions **(F1)(i)** and **(F2)(i)**.

Moreover, because $\tilde{b} = -b \ge -1$,

$$
f(b,y) - \tilde{b} \le 2 - \frac{1}{4}(1-\rho_0\epsilon)^2\lambda^2 + \frac{1}{2}\{(1-\rho_0\epsilon)^2 + 4 \parallel \phi_1 \parallel_\infty \epsilon\}\lambda + \xi_0\epsilon.
$$

Then there is a number $\lambda_1 > \lambda''$ such that $f(b,y) < b' < \tilde{\varphi}_*$ for all $\lambda \ge \lambda_1$. For such λ, f satisfies the condition **(F3)**. Thus, we have shown that there is a constant $\lambda_1 > 2$ such that f satisfies the condition **(F0)**$-$**(F3)** for all $\lambda > \lambda_1$ if ϵ satisfies (3.4).

Next, to check the condition **(F4)**, we must estimate g.

Lemma 3.1. *For any $v \in I \cup \tilde{I}$ and $y \in \Omega$,*

$$
\begin{aligned}
\parallel g(v,y) \parallel_2 &\le \lambda\frac{\kappa_1}{\kappa_2}\{M^{(1)} + (1+2K)\epsilon\}, \\
\parallel g(v,y) \parallel_\infty &\le \lambda\kappa_1 M_4\{2M^{(1)} + (1+2K)(\epsilon + K^{1-\frac{2}{p}}\epsilon^{\frac{2}{p}})\}
\end{aligned} \tag{3.5}
$$

with $p = 2$ for $d = 1,2,3$ and some $p > d/2$ for $d \ge 4$, where

$$
M^{(1)} = M^{(1)}(\lambda) = M_6(\lambda^2 + 8) \tag{3.6}
$$

with $M_6 = M_6(r_0, \Theta, D)$.

Proof. By Hölders's inequality, we get for any $p \geq 2$ and $y \in \Omega$,

$$\begin{aligned}
\| y \|_p &\leq \| y \|_\infty^{1-\frac{2}{p}} \| y \|_2^{\frac{2}{p}} \leq K^{1-\frac{2}{p}} \epsilon^{\frac{2}{p}}, \\
\| y^2 \|_p &= \| y \|_{2p}^2 \leq K^{2(1-\frac{1}{p})} \epsilon^{\frac{2}{p}}.
\end{aligned}$$

Recall the decomposition $u = z\phi_1 + y$ and $z = 1/(2\alpha_0) + mv$. Then, for $v \in I \cup \tilde{I}$,

$$\begin{aligned}
\| u^2 \|_p &\leq | z |^2 \| \phi_1^2 \|_p + 2 | z | \| \phi_1 y \|_p + \| y^2 \|_p \\
&\leq (| z | \| \phi_1 \|_{2p} + \| y \|_{2p})^2 \\
&\leq 2(| z |^2 \| \phi_1 \|_{2p}^2 + \| y \|_{2p}^2) \\
&\leq M^{(1)} + 2K^{2(1-\frac{1}{p})} \epsilon^{\frac{2}{p}},
\end{aligned}$$

where we have used

$$2 | z |^2 \| \phi_1 \|_{2p}^2 \leq 2(\frac{1}{2\alpha_0} + m)^2 \| \phi_1 \|_{2p}^2 \leq \frac{1}{4\alpha_0^2}(\lambda^2 + 8) M_3^2 = M^{(1)}$$

which comes from Lemma 2.3. Noting that

$$P_1^\perp G h(u) = P_1^\perp G y - P_1^\perp G u^2,$$

we conclude the lemma with the aid of Lemmas 2.2 and 2.4. ∎

Now we are ready to choose ϵ and K. Take a $\lambda_2 \geq \lambda_1$ and fix it. Recall $M^{(1)} = M^{(1)}(\lambda)$ given by (3.6). Now we put

$$\begin{aligned}
\epsilon &= 2\frac{1}{c_0} \lambda_2 M^{(1)}(\lambda_2) \epsilon_0, \\
K &= 3\lambda_2 M_4 M^{(1)}(\lambda_2) \epsilon_0,
\end{aligned}$$

with a small $\epsilon_0 > 0$ chosen so that

$$\begin{aligned}
&\epsilon \leq \frac{1}{a_0 + \rho_0}, \\
&(1 + 2K)\epsilon \leq M^{(1)}(\lambda_2), \\
&(1 + 2K)(\epsilon + K^{1-\frac{2}{p}} \epsilon^{\frac{2}{p}}) \leq M^{(1)}(\lambda_2),
\end{aligned} \tag{3.7}$$

can hold. By virtue of Lemma 2.1, there is a number $\delta_0 > 0$ such that

$$0 \leq \kappa_1 \leq \epsilon_0, \qquad \kappa_2 \geq c_0 > 0,$$

if $r + \| \theta \|_\infty \leq \delta_0$. Hence Lemma 3.1 yields

$$\begin{aligned}
\| g(v, y) \|_2 &\leq 2\frac{1}{c_0} \lambda_2 \epsilon_0 M^{(1)}(\lambda_2) = \epsilon, \\
\| g(v, y) \|_\infty &\leq 3\lambda_2 \epsilon_0 M_4 M^{(1)}(\lambda_2) = K,
\end{aligned}$$

which asserts that g fulfills **(F4)**. ∎

4. Proof of Theorem 1.2.

In this section, we will give a proof of Theorem 1.2. Let $F = (f, g)$ be as in Theorem 1.2 and let $\{(x_k, y_k) \in \boldsymbol{R} \times \Omega : k = 1, 2, \cdots, n\}$ be an n-periodic orbit of F. Then,

$$
(4.1) \qquad \begin{cases} (x_2, y_2) &= (f(x_1, y_1), g(x_1, y_1)), \\ &\vdots \\ (x_n, y_n) &= (f(x_{n-1}, y_{n-1}), g(x_{n-1}, y_{n-1})), \\ (x_1, y_1) &= (f(x_n, y_n), g(x_n, y_n)), \end{cases}
$$

which can be rewritten as

$$
(4.2) \qquad \begin{cases} x_1 &= \overbrace{f(f(\dots f(f}^{n}(x_1, y_1), y_2), \dots, y_n), \\ y_1 &= g(\overbrace{f(f(\dots f(f}^{n-1}(x_1, y_1), y_2), \dots, y_{n-1}), y_n), \\ y_2 &= g(x_1, y_1), \\ y_3 &= g(f(x_1, y_1), y_2), \\ \vdots & \\ y_n &= g(\overbrace{f(f(\dots f(f}^{n-2}(x_1, y_1), y_2), \dots, y_{n-2}), y_{n-1}). \end{cases}
$$

Set $\Omega^n = \overbrace{\Omega \times \Omega \times \cdots \times \Omega}^{n}$ and define the map $f_n : \boldsymbol{R} \times \Omega^n \to \boldsymbol{R}$ as

$$
(4.3) \qquad f_n(x, y^{(n)}) = \overbrace{f(f(\dots f(f}^{n}(x, y_1), y_2), \cdots, y_n),
$$

where $x \in \boldsymbol{R}$, $y_k \in \Omega$ and $y^{(n)} = (y_1, y_2, \cdots, y_n)^t \in \Omega^n$. Also, define $g_n : \boldsymbol{R} \times \Omega^n \to \Omega^n$ as

$$
(4.4) \qquad g_n(x, y^{(n)}) = \begin{pmatrix} g(f_{n-1}(x, y^{(n-1)}), y_n), \\ g(x, y_1), \\ g(f(x, y_1), y_2), \\ \vdots \\ g(f_{n-2}(x, y^{(n-2)}), y_{n-1}). \end{pmatrix}
$$

Then (4.2) can be written shortly as

$$
(4.5) \qquad \begin{cases} x_1 &= f_n(x_1, y^{(n)}), \\ y^{(n)} &= g_n(x_1, y^{(n)}). \end{cases}
$$

The following proposition is a key ingredient in proving Theorem 1.2.

Proposition 4.1. *Let f satisfy the conditions* **(F0)**–**(F3)**. *Then,*

(i) *for any $n = 1, 2, \cdots$,*

$$
(4.6) \qquad \exists 1\, p_n \in C(\Omega^n, \tilde{I}), \quad p_n(y^{(n)}) = f_n(p_n(y^{(n)}), y^{(n)}), \qquad (\forall y^{(n)} \in \Omega^n),
$$

(ii) *for any* $k = 1, 2, \cdots, n-1$,

$$f_k(p_n(y^{(n)}), y^{(k)}) \in I, \qquad (\forall y^{(k)} \in \Omega^k). \tag{4.7}$$

Since the proof of this proposition is rather lengthy, we defer it to the next section and here, we complete the proof of Theorem 1.2, admitting Proposition 4.1. Define the map $K_n : \Omega^n \to \Omega^n$ by

$$K_n(y^{(n)}) = g_n(p_n(y^{(n)}), y^{(n)}). \tag{4.8}$$

Since $p_n \in C(\Omega^n, \tilde{I})$, and since (4.7) holds, K_n is a compact map from Ω^n into itself, because so is g by the condition **(F4)**. Furthermore, Ω^n is a bounded, closed, convex set of Y^n. Therefore Schauder's fixed point theorem applies and K_n has a fixed point $\tilde{y}^{(n)} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) \in \Omega^n$, that is, $\tilde{y}^{(n)}$ satisfies

$$\begin{aligned} \tilde{y}_1 &= g(f_{n-1}(p_n(\tilde{y}^{(n)}), \tilde{y}^{(n-1)}), \tilde{y}_n), \\ \tilde{y}_2 &= g(p_n(\tilde{y}^{(n)}), \tilde{y}_1), \\ \tilde{y}_3 &= g(f(p_n(\tilde{y}^{(n)}), \tilde{y}_1), \tilde{y}_2), \\ &\vdots \\ \tilde{y}_n &= g(f_{n-2}(p_n(\tilde{y}^{(n)}), \tilde{y}^{(n-2)}), \tilde{y}_{n-1}). \end{aligned} \tag{4.9}$$

This and (4.6) show that $(p_n(\tilde{y}^{(n)}), \tilde{y}^{(n)}) \in \boldsymbol{R} \times \Omega^n$ solves (4.2). Therefore, $(p_n(\tilde{y}^{(n)}), \tilde{y}_1) \in \boldsymbol{R} \times \Omega$ is a candidate of n-periodic point of F. On the other hand, thanks to (4.7) and since $p_n(\tilde{y}^{(n)}) \in \tilde{I}$, it holds that, for all $k = 1, 2, \cdots, n-1$,

$$f_k(p_n(\tilde{y}^{(n)}), \tilde{y}^{(k)}) > 0 > p_n(\tilde{y}^{(n)})$$

holds, which implies that $(p_n(\tilde{y}^{(n)}), \tilde{y}_1)$ cannot be a k-periodic point of F for all $k = 1, 2, \cdots, n-1$. Thus we are done. ∎

5. Proofs of (1.16) and Proposition 4.1.

Proof of (1.16).

By the Intermediate Value Theorem and by **(F2)(i)**, there is a $\tilde{\varphi}(y) \in \tilde{I}$ such that $f(\tilde{\varphi}(y), y) = a$, $(\forall y \in \Omega)$. This $\tilde{\varphi}(y)$ is unique in $\tilde{I}$ by **(F2)(ii)**. To prove that $\tilde{\varphi}(y)$ is continuous, let $\{y_n\} \subset \Omega$ be a sequence satisfying $y_n \to y$, $(n \to \infty)$. We shall prove that $\tilde{\varphi}(y_n) \to \tilde{\varphi}(y)$. Suppose this be not the case. Then, there exists a subsequence $\{y_{n_k}\}$ such that $\mid \tilde{\varphi}(y_{n_k}) - \tilde{\varphi}(y) \mid > \varepsilon$ for all $k = 1, 2, \cdots$, with some $\varepsilon > 0$. Since $\tilde{I}$ is closed and bounded in $\boldsymbol{R}$, there exists a subsequence $\{y_{n_{k'}}\}$ and $\hat{\tilde{\varphi}} \neq \tilde{\varphi}(y) \in \tilde{I}$ such that $\tilde{\varphi}(y_{n_{k'}}) \to \hat{\tilde{\varphi}}$, $(k' \to \infty)$. Now by the definition of $\tilde{\varphi}$, we have $f(\tilde{\varphi}(y_{n_{k'}}), y_{n_{k'}}) = a$, and therefore $f(\hat{\tilde{\varphi}}, y) = a$ by **(F0)**. Hence, the uniqueness of $\tilde{\varphi}(y)$ implies that $\hat{\tilde{\varphi}} = \tilde{\varphi}(y)$, which is a contradiction. Thus, $\tilde{\varphi}$ is continuous in Ω. ∎

Proof of Proposition 4.1.

The proof of Proposition 4.1 will be based on the following lemma.

Lemma 5.1. *Let f be a map satisfying the conditions* **(F0)**$-$**(F3)**. *Let $\tilde{\varphi}_*$ be as in (1.17) and let f_n be a map defined by (4.3) from $\boldsymbol{R} \times \Omega^n$ into $\boldsymbol{R}$. Then for each even integer $n = 2k$, $k = 1, 2, \cdots$, the following holds.*

$(\mathbf{H0})_n$ $f_n, f_{n,x} \in C(\boldsymbol{R} \times \Omega^n, \boldsymbol{R})$.

$(\mathbf{H1})_n$ *There are two maps c_n and $d_n \in C(\Omega^n, I)$ satisfying the following.*

(i) $a < c_n(u) < d_n(u) < b$, $(\forall u \in \Omega^n)$.

(ii) $f_n(c_n(u), u) \leq \tilde{\varphi}_*$, $\quad f_n(d_n(u), u) \geq b$, $\qquad (\forall u \in \Omega^n)$,

(iii) $f_{n,x}(x, u) \geq 0$, $\qquad (\forall u \in \Omega^n,\ x \in I_n(u))$,

where

$$I_n(u) = [c_n(u), d_n(u)] \subset I. \tag{5.1}$$

$(\mathbf{H2})_n$ *There are two maps $\tilde{c}_n$ and $\tilde{d}_n \in C(\Omega^n, \tilde{I})$ satisfying the following.*

(i) $\tilde{b} < \tilde{d}_n(u) < \tilde{c}_n(u) < \tilde{a}$, $(\forall u \in \Omega^n)$.

(ii) $f_n(\tilde{c}_n(u), u) \leq \tilde{\varphi}_*$, $\quad f_n(\tilde{d}_n(u), u) \geq b$, $\qquad (\forall u \in \Omega^n)$,

(iii) $f_{n,x}(x, u) \leq 0$, $\qquad (\forall u \in \Omega^n,\ x \in \tilde{I}_n(u))$,

where

$$\tilde{I}_n(u) = [\tilde{d}_n(u), \tilde{c}_n(u)] \subset \tilde{I}. \tag{5.2}$$

Proof. We prove this lemma by induction. First, consider the case $n = 2$ and set $u = (y_1, y_2) \in \Omega^2$. It is clear that f_2 satisfies the condition $(\mathbf{H0})_2$. By the Intermediate Value Theorem and by **(F1)**, for each $y \in \Omega$, there exist $\phi(y)$, $\varphi(y) \in I$ such that

$$\begin{aligned} f(\phi(y), y) &= b, \qquad (\forall y \in \Omega), \\ f(\varphi(y), y) &= a, \qquad (\forall y \in \Omega). \end{aligned}$$

These $\phi(y)$, $\varphi(y)$ are unique in I and that ϕ, $\varphi \in C(\Omega, I)$ can be proved in the same way as in the proof of (1.16). Note that by the condition **(F1)(ii)**, $\phi(y) < \varphi(y)$ for all $y \in \Omega$. Then the conditions **(F1)(i)** and **(F3)** imply that

$$\begin{aligned} f_2(\phi(y_1), u) &= f(f(\phi(y_1), y_1), y_2) = f(b, y_2) \leq \tilde{\varphi}_*, \qquad (\forall u \in \Omega^2), \\ f_2(\varphi(y_1), u) &= f(f(\varphi(y_1), y_1), y_2) = f(a, y_2) \geq b, \qquad (\forall u \in \Omega^2). \end{aligned} \tag{5.3}$$

On the other hand, it is clear that $f(x, y) \in I$ if $y \in \Omega$ and $x \in [\phi(y), \varphi(y)] \subset I$, and therefore it holds that

$$f_{2,x}(x, u) = f_x(x, y_1) f_x(f(x, y_1), y_2) \geq 0, \quad (\forall u \in \Omega^2, x \in [\phi(y_1), \varphi(y_1)]). \tag{5.4}$$

Now, putting

$$\begin{aligned} c_2(u) &= \phi(y_1), \\ d_2(u) &= \varphi(y_1) \end{aligned}$$

will prove $(\mathbf{H1})_2$.

Similarly, we can show the existence of a unique map $\tilde{\phi} \in C(\Omega, \tilde{I})$ which satisfies

$$f(\tilde{\phi}(y), y) = b, \qquad (\forall y \in \Omega).$$

Then, with $\tilde{c}_2(u) = \tilde{\phi}(y_1)$ and $\tilde{d}_2(u) = \tilde{\varphi}(y_1)$ where $\tilde{\varphi}$ is as in (1.16), we see that $\mathbf{(H2)}_2$ is fulfilled as well. Thus we proved the lemma for $n = 2$.

Next, assume that $\mathbf{(H0)}_n \sim \mathbf{(H2)}_n$ are true for some $n = 2k$. In the sequel, we use the notation

$$w = (u, z) \in \Omega^n \times \Omega^2 = \Omega^{n+2}.$$

Since by definition (4.3),

$$f_{n+2}(x, w) = f_2(f_n(x, u), z)$$

holds, it follows from $\mathbf{(H0)}_2$ and $\mathbf{(H0)}_n$ that f_{n+2} satisfies the condition $\mathbf{(H0)}_{n+2}$. Again by the Intermediate Value Theorem, and from $\mathbf{(H1)}_n$, there exist $\phi(w)$, $\varphi(w)$ $(\forall w \in \Omega^{n+2})$ such that for any $w = (u, z) \in \Omega^{n+2}$,

$$\begin{aligned} &\phi(w), \varphi(w) \in I_n(u), \\ &f_n(\phi(w), u) = c_2(z), \\ &f_n(\varphi(w), u) = d_2(z). \end{aligned} \tag{5.5}$$

Similarly by $\mathbf{(H2)}_n$, there are $\tilde{\phi}(w)$, $\tilde{\varphi}(w)$ $(\forall w \in \Omega^{n+2})$ such that such that

$$\begin{aligned} &\tilde{\phi}(w), \tilde{\varphi}(w) \in \tilde{I}_n(u), \\ &f_n(\tilde{\phi}(w), u) = c_2(z), \\ &f_n(\tilde{\varphi}(w), u) = d_2(z). \end{aligned} \tag{5.6}$$

As before, ϕ, φ are unique and in $C(\Omega^n, I)$ while $\tilde{\phi}$, $\tilde{\varphi}$ are unique and in $C(\Omega^n, \tilde{I})$. By the condition $\mathbf{(H1)}_2\mathbf{(ii)}$, we have

$$\begin{aligned} f_{n+2}(\phi(w), w) &= f_2(f_n(\phi(w), u), z) \\ &= f_2(c_2(z), z) \\ &\leq \tilde{\varphi}_*, \end{aligned}$$

and also

$$\begin{aligned} f_{n+2}(\varphi(w), w) &= f_2(f_n(\varphi(w), u), z) \\ &= f_2(d_2(z), z) \\ &\geq b. \end{aligned}$$

Similarly, it holds that

$$\begin{aligned} f_{n+2}(\tilde{\phi}(w), w) &\leq \tilde{\varphi}_*, \\ f_{n+2}(\tilde{\varphi}(w), w) &\geq b. \end{aligned} \tag{5.7}$$

As a consequence, f_{n+2} satisfies the conditions $\mathbf{(H1)}_{n+2}$ and $\mathbf{(H2)}_{n+2}$ if we set

$$\begin{aligned} c_{n+2}(w) &= \phi(w), \\ d_{n+2}(w) &= \varphi(w), \\ \tilde{c}_{n+2}(w) &= \tilde{\phi}(w), \\ \tilde{d}_{n+2}(w) &= \tilde{\varphi}(w). \end{aligned} \tag{5.8}$$

Note that

$$
(5.9)\qquad \begin{aligned} I_{n+2}(w) &= [c_{n+2}(w), d_{n+2}(w)] \subset I_n(u), \\ \tilde{I}_{n+2}(w) &= [\tilde{d}_{n+2}(w), \tilde{c}_{n+2}(w)] \subset \tilde{I}_n(u). \end{aligned}
$$

Moreover, from (5.5), (5.6) and the conditions $\mathbf{(H1)}_n\mathbf{(iii)}$, $\mathbf{(H2)}_n\mathbf{(iii)}$, it follows that

$$
f_n(x,u) \in I_2(z), \qquad (\forall w \in \Omega^{n+2}, \quad x \in \tilde{I}_{n+2}(w) \cup I_{n+2}(w)).
$$

So, by $\mathbf{(H1)}_2\mathbf{(iii)}$, $\mathbf{(H2)}_2\mathbf{(iii)}$, $\mathbf{(H1)}_n\mathbf{(iii)}$, $\mathbf{(H2)}_n\mathbf{(iii)}$, (5.1), (5.2) and (5.9), it follows that

$$
f_{n+2,x}(x,w) = f_{n,x}(x,u) f_{2,x}(f_n(x,u),z) \geq 0, \qquad (\forall w \in \Omega^{n+2}, \quad x \in I_{n+2}(w)),
$$

and that

$$
f_{n+2,x}(x,w) = f_{n,x}(x,u) f_{2,x}(f_n(x,u),z) \leq 0, \qquad (\forall w \in \Omega^{n+2}, \quad x \in \tilde{I}_{n+2}(w)).
$$

This completes the proof of Lemma 5.1. ■

Proof of Proposition 4.1(i).

We continue to use the same notations as in the proof of Lemma 5.1. First, we consider the case of $n = 2k$, $(k = 1, 2, 3, \cdots)$. From the condition $\mathbf{(H2)}_n\mathbf{(ii)}$, (5.6) and (5.8), we have

$$
\begin{aligned} f_n(\tilde{c}_n(u), u) &\leq \tilde{\varphi}^* < \tilde{b} < \tilde{c}_n(u), \\ f_n(\tilde{c}_{n+2}(w), u) &= c_2(z) > 0. \end{aligned}
$$

Then, there is a unique

$$
(5.10)\qquad p_n(u) \in (\tilde{c}_{n+2}(w), \tilde{c}_n(u)) \subset \tilde{I}_n(u)
$$

such that

$$
p_n(u) = f_n(p_n(u), u)
$$

for all $u \in \Omega^n$, by the Intermediate Value Theorem and $\mathbf{(H2)}_n\mathbf{(ii)},\mathbf{(iii)}$. Also, by the contradiction argument as in the proof of (1.16), we can show that p_n is continuous in Ω^n.

Next, we consider the case $n = 2k+1$ $(k = 1, 2, 3, \cdots)$. By (5.6) and the definition of c_2, and with the notation $w = (v, y) \in \Omega^{2k+1} \times \Omega$,

$$
(5.11)\qquad \begin{aligned} f_{2k+1}(\tilde{c}_{2k+2}(w), v) &= f(f_{2k}(\tilde{c}_{2k+2}(w), u), y) \\ &= f(\tilde{c}_2(z), y) \\ &= f(\tilde{\phi}(y), y) = b > 0. \end{aligned}
$$

Also, by $\mathbf{(H2)}_{2k}\mathbf{(ii)}$, (5.6) and (5.8), there is a $\tilde{d}_{2k+1}(v) \in [\tilde{d}_{2k}(u), \tilde{d}_{2k+2}(w))$ such that

$$
(5.12)\qquad f_{2k}(\tilde{d}_{2k+1}(v), u) = b.
$$

Then we can see that

$$
(5.13)\qquad \begin{aligned} f_{2k+1}(\tilde{d}_{2k+1}(v), v) &= f(f_{2k}(\tilde{d}_{2k+1}(v), u), y) \\ &= f(b, y) \leq \tilde{\varphi}_* < \tilde{b} < \tilde{d}_{2k+1}(v). \end{aligned}
$$

In addition, we see that

$$f_{2k+1,x}(x,v) = f_{2k,x}(x,u) f_x(f_{2k}(x,u),y) \geq 0, \tag{5.14}$$

for all $x \in [\tilde{d}_{2k+1}(v), \tilde{c}_{2k+2}(w)] \subset \tilde{I}_{2k}(u)$, because $f_{2k}(x,u) \in [c_2(z), b] \subset I$ for such x. Then (5.11), (5.13) and (5.14) imply that there exists a unique

$$p_{2k+1}(v) \in (\tilde{d}_{2k+1}(v), \tilde{c}_{2k+2}(w)) \subset \tilde{I}_{2k}(u) \tag{5.15}$$

satisfying

$$p_{2k+1}(v) = f_{2k+1}(p_{2k+1}(v), v), \qquad (\forall v \in \Omega^{2k+1}).$$

The continuity of p_{2k+1} in Ω^{2k+1} can be proved by the same way as (1.16). ∎

Proof of Proposition 4.1(ii).

We again use the same notation as in the proof of Lemma 5.1, that is, $w = (u, z) \in \Omega^{2k} \times \Omega^2$. Also, we will use the notation $y^{(n)} = (y_1, y_2, \cdots, y_n)$. Note that $u = y^{(2k)}$. First, we consider the case $n = 2k$. Since (5.10) holds, by (5.5), (5.9) and $\mathbf{(H1)}_m\mathbf{(iii)}$, we have

$$f_m(p_n(u), y^{(m)}) \in [c_2(z), d_2(z)] \subset I, \qquad \forall m = 2, 4, 6, \cdots, n-2. \tag{5.16}$$

Also, by the condition **(F1)(ii)** and (5.16), we see that, for all $m = 2, 4, 6, \cdots, n-2$,

$$\begin{aligned} f_{m+1}(p_n(u), y^{(m+1)}) &= f(f_m(p_n(u), y^{(m)}), y) \\ &\in [f(d_2(z), y), f(c_2(z), y)] \\ &= [a, b] = I. \end{aligned}$$

and, it holds that

$$f(p_n(u), y) \in [f(\tilde{d}_2(z), y), f(\tilde{c}_2(z), y)] = [a, b] = I,$$

because $p_n(u) \in \tilde{I}_n(u) \subset \tilde{I}_2(z)$.

Next, we consider the case $n = 2k + 1$. Since (5.15) holds, we can show similarly as above that

$$f_m(p_{2k+1}(v), y^{(m)}) \in I, \qquad (m = 1, 2, \cdots 2k - 1).$$

Moreover, by (5.6), (5.12), (5.15), and $\mathbf{(H2)}_{2k}\mathbf{(iii)}$,

$$f_{2k}(p_{2k+1}(v), w) \in [f_{2k}(\tilde{c}_{2k+2}(u), w), f_{2k}(\tilde{d}_{2k+1}(v), w)] = [c_2(z), b] \subset I.$$

Thus we are done. ∎

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